Available online at www.sciencedirect.com
SCIENCE DIRECT
ELSEVIER
Journal of APPLIED MATHEMATICS AND MECHANICS
www.elsevier.com/locate/jappmathmech
Journal of Applied Mathematics and Mechanics 69 (2005) 832–836

LINEAR INTEGRALS OF NON-HOLONOMIC SYSTEMS WITH NON-LINEAR CONSTRAINTS†

D. N. ZEKOVICH

Belgrade

(*Received* 25 *February* 2005)

The problem of the existence of linear integrals of the equation of motion of a mechanical system, subjected to non-linear constraints, is considered. Existing results for holonomic systems, and also for non-holonomic systems with linear constraints, are extended to systems with non-linear non-holonomic constraints. Examples are given. © 2006 Elsevier Ltd. All rights reserved.

1. FORMULATION OF THE PROBLEM

Consider a mechanical system, whose position is defined by the generalized coordinates $q^1, \dots, q^n$; the kinetic energy is given by the expression $T = a_{ij}\dot{q}^i\dot{q}^j/2$, while the potential energy $\Pi = \Pi(q^i)$. Here and henceforth summation is carried out over repeated indices. The indices take the following values: $i, j, k, s = 1, \dots, n$; $\alpha, \beta = 1, \dots, m$; $\nu, \rho = 1 + m, \dots, m + l = n$. The system is subject to non-linear non-holonomic constraints of the form

$$f^\nu(q^i, \dot{q}^i) = 0 \tag{1.1}$$

We will write the equations of this system in the Hamiltonian form

$$\dot{q}^j = \frac{\partial H}{\partial p_j}, \quad \dot{p}_j = -\frac{\partial H}{\partial q^j} + \lambda_\nu \frac{\partial f^\nu}{\partial \dot{q}^j} \tag{1.2}$$

Here $p_j = \partial T/\partial \dot{q}^j$, H is Hamilton's function, which has the form $H = b^{ij}p_i/2 + \Pi$, b^{ij} are the elements of the matrix inverse to $\|a_{ij}\|$ and λ_ν are multipliers of the constraints.

We will assume that the equations of motion (1.2) have an integral that is linear with respect to the momenta:

$$\varphi = \varepsilon^j p_j = \text{const} \tag{1.3}$$

The necessary and sufficient conditions for linear integrals of the equations of motion with linear non-holonomic constraints of the form $c_j^\nu \dot{q}^j = 0$ to exist were obtained in [1, 2]: the function (1.3) is the integral of the equations of motion if and only if (a) $c_j^\nu \varepsilon^j = 0$ and (b) (1.3) is the integral of the equations $\dot{q}^j = \partial H/\partial p_j, \dot{p}_j = -\partial H/\partial q^j$. However, it was shown in [3] that in [1, 2] there is in fact no analysis of the necessity of condition (a) and (b), and it was shown by examples that conditions (a) and (b) are not necessary, i.e. a case exists when $c_j^\nu \varepsilon^j \neq 0$, but the system has a linear integral. Using the generalized idea of a "cyclic" coordinate, a theorem on linear integrals was proved in [3, 4], similar to the theorem for holonomic mechanical systems [5–7].

†*Prikl. Mat. Mekh.* Vol. 69, No. 6, pp. 929–934, 2005.

0021-8928/$—see front matter.
doi: 10.1016/j.jappmathmech.2005.11.004

2. CONDITIONS FOR LINEAR INTEGRALS TO EXIST

Using methods described previously in [1–4], we will extend these results to mechanical systems with non-linear non-holonomic constraints [1.1].

A coordinate q^n, for which

$$\frac{\partial H}{\partial q^n} = 0, \quad \lambda_\nu \frac{\partial f^\nu}{\partial \dot{q}^n} = 0$$

is usually called a cyclic coordinate. Note a special case of this definition

$$\frac{\partial H}{\partial q^n} = 0, \quad \frac{\partial f^{m+1}}{\partial \dot{q}^n} = 0, \ldots, \frac{\partial f^{m+l}}{\partial \dot{q}^n} = 0$$

Constraint (1.1) can be rewritten in the form

$$F^\nu(q^i, p_i) = 0 \tag{2.1}$$

Further, we will mean by an expression in square brackets and expression in which 1 momenta have been eliminated using constraint (2.1).

It is obvious that the expression

$$\gamma = k_\nu F^\nu \tag{2.2}$$

where k_ν are arbitrary functions of the coordinates, is a linear integral of Eqs (1.2), if the initial conditions are chosen in accordance with the constraint equations. We will call the coordinate q^n "cyclic" if

$$\left[\frac{\partial H}{\partial q^n}\right] = 0 \ \left(\text{in particular, } \frac{\partial H}{\partial q^n} = 0\right), \quad \frac{\partial f^{m+1}}{\partial \dot{q}^n} = 0, \ldots, \frac{\partial f^{m+l}}{\partial \dot{q}^n} = 0$$

We have the following assertion: only those systems possess integrals which are linear in the momenta which either have "cyclic" coordinates or can be converted into systems with "cyclic" coordinates by means of an extended point transformation (compare with the similar assertion in [4]).

In fact, the expression

$$\psi = \varphi - k_\nu F^\nu = \eta^j p_j$$

where k_ν is the solution of the algebraic equations

$$k_\nu \frac{\partial F^\nu}{\partial p_i} \frac{\partial f^\rho}{\partial \dot{q}^i} = \varepsilon^j \frac{\partial f^\rho}{\partial \dot{q}^j}$$

is an integral of Eqs (1.2) on the set of real trajectories of the system, where

$$\frac{\partial f^\rho}{\partial \dot{q}^j} \eta^j = 0 \tag{2.3}$$

Consider the system of equations

$$\frac{\partial q^1}{\partial \eta^1} = \ldots = \frac{\partial q^n}{\partial \eta^n} \tag{2.4}$$

We will assume that at least one of the functions $\eta^1, \ldots, \eta^n$ is non-zero, for example $\eta^1 \neq 0$. Suppose the system of solution of Eqs (2.4) consists of $n - 1$ integrals $Q^r(q^1, \ldots, q^n) = \text{const}$ $(r = 1, \ldots, n - 1)$ and suppose the function Q^n is defined by the equation

$$Q^n = \int \frac{dq^1}{\eta^1}$$

where, in the expression for η^1 the coordinates $q^2, \ldots, q^n$ must be expressed in terms of $q^1, Q^1, \ldots, Q^{n-1}$. If the variables are changed in such a way that only the quantity Q^n is changed, while the quantities $Q^1, \ldots, Q^{n-1}$ remain constant, then, by virtue of the previous equation, we obtain

$$\frac{dq^1}{\eta^1} = \ldots = \frac{dq^n}{\eta^n} = dQ^n \tag{2.5}$$

If the quantities $Q^1, \ldots, Q^n$ are considered as new variables, in terms of which one can express the previous variables $q^1, \ldots, q^n$, then

$$\frac{\partial q^s}{\partial Q^n} = \eta^s$$

We will now consider an extended point transformation from the variables $q^1, \ldots, q^n$ to the variables $Q^1, \ldots, Q^n$, so that the new momenta $P_1, \ldots, P_n$ are defined by the equations

$$P_k = p_s \frac{\partial q^s}{\partial Q^k}$$

(see [5, 6]). As a result of this transformation, Eqs (1.2) take the following form [2]

$$\dot{Q}^j = \frac{\partial \bar{H}}{\partial P_j}, \quad \dot{P}_j = -\frac{\partial \bar{H}}{\partial Q_j} + \lambda_\nu A_j^\nu; \quad A_j^\nu = \frac{\partial f^\nu}{\partial \dot{q}^i} \frac{\partial q^i}{\partial Q^j}$$

where $\bar{H}$ is the Hamilton function of the system, represented in the new variables.

The integral ψ = const is converted into the integral P_n = const. Since $[\dot{P}_n] = 0$ and $A_n^{m+1} = 0, \ldots, A_n^n = 0$ (see relations (2.3) and (2.5)), we have $[\partial \bar{H}/\partial Q^n] = 0$. This means that the coordinate Q^n is "cyclic".

3. EXAMPLES

Example 1. Suppose two heavy point masses M_1 and M_2 of unit mass, connected by a rod of fixed length $2l$, move in a vertical plane in such a way that their velocities $\mathbf{v}_1$ and $\mathbf{v}_2$ are parallel. This constraint can be achieved in practice by attaching a knife edge at the point M perpendicular to the direction M_1M_2 [8] (see Fig. 1).

Suppose x_1, y_1 and x_2, y_2 are the coordinates of the points M_1 and M_2. The equations of the constraints of the system can be written in the form

$$f^3 = (x_2 - x_1)^2 + (y_2 - y_1)^2 - (2l)^2 = 0, \quad f^4 = \dot{x}_1\dot{y}_2 - \dot{x}_2\dot{y}_1 = 0$$

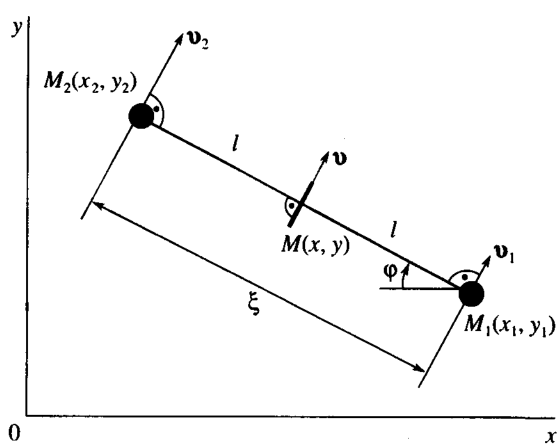

Fig. 1

where the second equation expresses the condition that $\mathbf{v}_1$ and $\mathbf{v}_2$ must be parallel and represents a non-linear non-holonomic constraint. The Lagrange function of the system and the Hamilton function have the form

$$L = \frac{1}{2}(\dot{x}_1^2 + \dot{y}_1^2 + \dot{x}_2^2 + \dot{y}_2^2) - g(y_1 + y_2), \quad H = \frac{1}{2}(p_1^2 + p_2^2 + p_3^2 + p_4^2) + g(y_1 + y_2) \tag{3.1}$$

We will define new variables by the following relations

$$Q^1 = \frac{q^1 + q^3}{2}, \quad Q^2 = \frac{q^2 + q^4}{2}, \quad Q^3 = \sqrt{(q^3 - q^1)^2 + (q^4 - q^2)^2}, \quad Q^4 = \int \frac{dq^1}{\eta^1} = \int \frac{dq^1}{q^4 - q^2}$$
$$q^1 = x_1, \quad q^2 = y_1, \quad q^3 = x_2, \quad q^4 = y_2 \tag{3.2}$$

where q^4 and q^2 in the last integral must be expressed in terms of q^1, Q^1, Q^2 and Q^3. The formulae which express the old variables in terms of the new ones have the form

$$q^{1,3} = Q^1 \mp \frac{Q^3}{2}\cos Q^4, \quad q^{2,4} = Q^2 \mp \frac{Q^3}{2}\sin Q^4$$

The meaning of the new variables is clear: Q^1 and Q^2 are the coordinates x and y of the centre of the rod, Q^3 is a constant quantity, equal to the length of the rod and Q^4 is the angle of inclination of the rod to the x axis.

The equations of motion allow of the following integral, which is linear and homogeneous in the momenta,

$$(x_2 - x_1)(\dot{y}_2 - \dot{y}_1) - (y_2 - y_1)(\dot{x}_2 - \dot{x}_1) = \text{const} \tag{3.3}$$

The Lagrange function and the equations of the constraints of the converted system, and also the Hamilton function can be written in the form

$$\bar{L} = (\dot{Q}^1)^2 + (\dot{Q}_2)^2 + (Q^3/2)(\dot{Q}^4)^2 - 2gQ^2$$
$$\bar{f}^3 = \dot{Q}^3 = 0, \quad \bar{f}^4 = \dot{Q}^1 - \dot{Q}^2 \operatorname{tg} Q^4 = 0$$
$$\bar{H} = \frac{1}{4}P_1^2 + \frac{1}{4}P_2^2 + \frac{1}{(Q^3)^2}P_4^2 + 2gQ^2$$

Since

$$\partial\bar{H}/\partial Q^4 = 0, \quad A_4^3 = 0, \quad A_4^4 = 0 \tag{3.4}$$

the coordinate Q^4 is cyclic, and hence we have the integral $P_4 = \frac{1}{2}(Q^3)^2\dot{Q}^4 = \text{const}$, i.e. $\dot{\varphi} = \text{const}$.

Example 2. A system consists of two point masses M_1 and M_2 of unit mass, connected by a weightless structure (a "fork"), as shown in Fig. 2. There are two knife edges at the points M_1 and M_2, one of which is parallel and the other perpendicular to the section M_1M_2. The system is in a potential force field, the potential energy Π of which depends only on the distance between the points M_1 and M_2. For example, these points can be connected by an elastic spring [8].

The equations of the constraints of the system can be written in the form

$$f^3 = \dot{x}_1\cos\varphi + \dot{y}_1\sin\varphi = 0, \quad f^4 = \dot{x}_1\dot{x}_2 + \dot{y}_1\dot{y}_2 = 0$$

where the second equation expresses the condition that the velocities $\mathbf{v}_1$ and $\mathbf{v}_2$ are orthogonal and represents a non-linear non-holonomic constraint.

The Lagrange and Hamilton functions have a form which differs from (3.1) solely in that the function $g(y_1 + y_2)$ is replaced by $\Pi(x_1, y_1, x_2, y_2)$.

We defined the new variables by relations similar to (3.2), with the exception that now $Q^1 = q^3$ and $Q^2 = q^4$. The formulae that express the old variables in terms of the new ones have the form

$$q^1 = Q^1 - Q^3\cos Q^4, \quad q^2 = Q^2 - Q^3\sin Q^4, \quad q^3 = Q^1, \quad q^4 = Q^2$$

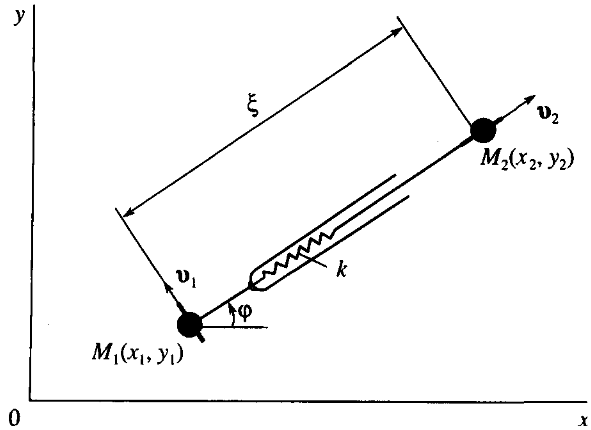

Fig. 2

The meaning of the new variables is clear: Q^1 and Q^2 are the coordinates of the point M_2, Q^3 is the length of the section M_1M_2, and Q^4 is the angle of inclination of M_1M_2 to the x axis.

The equations of motion of the system allow of an integral of the form (3.3) that is linear and homogeneous in the momenta.

The Lagrange function and the equations of the constraints of the transformed system, and also Hamilton's function have the form

$$\bar{L} = \frac{1}{2}[(\dot{Q}^1)^2 + (\dot{Q}^2)^2] + \frac{1}{2}(Q^3\dot{Q}^4)^2 - \frac{1}{2}k(Q^3 - Q_0^3)^2$$

$$\bar{f}^3 = \dot{Q}^1\cos Q^4 + \dot{Q}^2\sin Q^4 - \dot{Q}^3 = 0, \quad \bar{f}^4 = (\dot{Q}^1)^2 + (\dot{Q}^2)^2 - (\dot{Q}^3)^2 = 0$$

$$\bar{H} = \frac{1}{2}P_1^2 + \frac{1}{2}P_2^2 + \frac{1}{2(Q^3)^2}P_4^2 + \frac{1}{2}k(Q^3 - Q_0^3)^2$$

Since conditions (3.4) are satisfied, the coordinate Q^4 is cyclic, and hence we have the cyclic integral $P_4 = (Q^3)^2\dot{Q}^4 = \text{const}$, i.e. $\xi^2\dot{\varphi} = \text{const}$.

This research was supported by the Ministry of Science and Technology of Serbia (1616) with the participation of the Mathematical Institute SANU.

REFERENCES

1. AGOSTINELLI, C., Nuova forma sintetica delle equazioni del moto di un sistema anolonome ed esistenza di un integrale lineare nelle velosita lagrangiane. *Boll. Unione mat. Ital.* Ser3. 1956, **11**, 1, 1–9.
2. NAZIYEV, E. Kh., Mechanical systems with integrals that are linear in the momenta. *Vestnik MGU. Ser. 1. Matematika, Mekhanika*, 1969, 2, 77–85.
3. SUMBATOV, A. S., Linear integrals of the equations of motion with multipliers of the constraints. *Vestnik MGU. Ser. 1. Matematika, Mekhanika*, 1971, 4, 99–101.
4. SUMBATOV, A. S., Linear integrals of non-holonomic systems. *Vestnik MGU. Ser. 1. Matematika, Mekhanika*, 1972, 6, 77–83.
5. WHITTAKER, E. T., *A Treatise on Analytical Dynamics*. University Press, Cambridge, 1927.
6. PARS, L. A., *A Treatise on Analytical Dynamics*. Heinemann, London, 1965.
7. ILIYEV, I. O., Linear integrals of a holonomic mechanical system. *Prikl. Mat. Mekh.*, 1970, **34**, 4, 751–755.
8. ZEKOVICH, D., Examples of non-linear non-holonomic constraints in classical mechanics. *Vestnik MGU. Ser. 1. Matematika, Mekhanika*, 1991, 1, 100–103.

Translated by R.C.G.